

Computing the density of Ricci-solitons on $\mathbb{CP}^2 \# 2\overline{\mathbb{CP}^2}$

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Abstract

This is a short note explaining how one can compute the Gaussian density of the Kähler-Ricci soliton and the conformally Kähler, Einstein metric on the two point blow-up of the complex projective plane.

1 Introduction

We begin by recalling the notion of Ricci-soliton.

Definition 1 *A Riemannian metric g is called a Ricci-soliton if there exists a vector field V and constant ρ satisfying*

$$Ric(g) + L_V g = \rho g.$$

For $\rho > 0$ the soliton is called shrinking, for $\rho = 0$ the soliton is called steady and for $\rho < 0$ the soliton is called expanding. Moreover, if V is the gradient of a smooth function f , the soliton is called a gradient Ricci-soliton with potential function f .

We see that by setting $V = 0$ in the above definition we recover the notion of an Einstein metric so solitons can be thought of as generalisations of Einstein metrics. Ricci-Solitons are particularly important in the theory of the Ricci-flow. If one evolves a one-parameter family of metrics $g(t)$ via the Ricci-flow

$$\frac{\partial g}{\partial t} = -2Ric(g),$$

then Ricci-solitons represent fixed points of the flow up to diffeomorphism. In [1] H. Cao lists the Gaussian density (defined by Cao, Hamilton and Ilmanen in [2]) of various shrinking Ricci-solitons on compact complex surfaces. Some values omitted from the list were that of the Kähler-Ricci soliton on $\mathbb{CP}^2 \# 2\overline{\mathbb{CP}^2}$ (the existence of which is due to Wang and Zhu [11]) and that of the Einstein metric, also on this manifold, shown to exist by Chen, LeBrun and Weber in [4]. The former metric is not known in any explicit form making it hard to compute

geometric quantities whilst the later has only recently been shown to even exist and is also not known explicitly.

Recently, good progress has been made in approximating distinguished metrics numerically and in [6] Headrick and Wiseman use their approximation to the Kähler-Ricci soliton to give a value of its density. One feature they exploit is that the soliton is a ‘toric-Kähler’ metric and so approximating the metric is equivalent to approximating a certain class of convex functions on a pentagon. Various authors ([11], [5]) use this framework to develop the existence theory of Kähler-Ricci solitons. In this note we exploit this framework and show how the density of these two metrics can be computed by elementary means and without any explicit knowledge of the metric itself.

The two metrics discussed above are analogous to a pair of metrics on $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$. The first is an Einstein metric explicitly constructed by Page [9] and the second is a Kähler-Ricci soliton constructed independently by Koiso and Cao [7], [3]. The Gaussian densities of these metrics is already known, however, using our method, one can easily verify the values given in [1]. The updated portion of the table is as follows:

Manifold	Metric Name	Type	Gaussian Density Θ
$\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$	Koiso-Cao Soliton	Kähler-Ricci Soliton	0.5179
$\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$	Page metric	Einstein	0.5172
$\mathbb{CP}^2 \# 2\overline{\mathbb{CP}^2}$	Chen-LeBrun-Weber metric	Einstein	0.4552
$\mathbb{CP}^2 \# 2\overline{\mathbb{CP}^2}$	Wang-Zhu Soliton	Kähler-Ricci Soliton	0.4549

The calculation of these numbers is useful when considering the behaviour of the metrics when evolved via the Ricci-flow. A theorem of Perelman [10] states that the Gaussian density must increase under the flow and can only be stationary at a soliton metric. Therefore one would not be able to perturb the Chen-LeBrun-Weber metric and then flow to the Wang-Zhu soliton. It is interesting that this is not necessarily the case for $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ as here the soliton has the higher density. It would be intriguing if there was some non-numerical explanation for this asymmetry. It is also slightly remarkable just how close the numbers are for each manifold. This underlines the importance of the calculation of these values rather than just the estimation of them.

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2 Calculation

2.1 The density of Einstein metrics

We begin by recalling the definition of the Perelman \mathcal{W} -functional, ν -energy and Gaussian density Θ .

Definition 2 (\mathcal{W} -functional) *If (M^n, g) is a closed Riemannian manifold, f a smooth function on M and $\tau > 0$ a real number then we define*

$$\mathcal{W}(g, f, \tau) = \int_M [\tau(R + |\nabla f|^2) + f - n](4\pi\tau)^{-\frac{n}{2}} e^{-f} dV_g$$

where R is the scalar curvature of g .

Definition 3 (ν -energy and Gaussian Density) *Let M^n , g , f and τ be as above, then*

$$\nu(g) = \inf \left\{ \mathcal{W}(g, f, \tau) : \frac{1}{(4\pi\tau)^{\frac{n}{2}}} \int_M e^{-f} dV = 1 \right\}.$$

The Gaussian density of g , $\Theta(g)$, is

$$\Theta = e^{\nu(g)}.$$

Note that the ν -energy is invariant under re-scalings of the metric.

Ricci-solitons are stationary points of the ν -energy and if the soliton happens to be a trivial one, i.e. an Einstein metric, then one has the following formula for the value of Θ . Note that we require positive scalar curvature.

Proposition 1 (See [1] for example) *If (M^n, g) is an Einstein manifold of positive scalar curvature then*

$$\Theta(g) = \left(\frac{R}{2\pi n e} \right)^{\frac{n}{2}} \text{Vol}(M, g)$$

where R is the scalar curvature of g and $\text{Vol}(M, g)$ is the volume of M with respect to g .

So we see that we only require the value of the scalar curvature and the volume of the manifold in order to compute the density.

2.2 Conformally Kähler, Einstein Metrics

The Chen-LeBrun-Weber metric g_{clw} is constructed by considering extremal Kähler metrics. For a fixed Kähler class $[\omega]$ on a complex manifold M one defines the Calabi energy of a representative metric g as

$$Ca(g) = \int_M R^2(g) dV_g$$

where $R(g)$ is the scalar curvature of g . A metric g is called extremal if it minimises the Calabi energy over all other Kähler metrics in the class $[\omega]$ (note that such a minimising representative does not always exist). Chen, LeBrun and Weber show that there is a Kähler class on $\mathbb{CP}^2 \# 2\overline{\mathbb{CP}^2}$ that admits an extremal metric, g_{ext} say, for which the conformally related metric $g_{clw} := R^{-2}(g_{ext})g_{ext}$ is Einstein.

2.2.1 The value of $\Theta(g_{clw})$

In the following, M is a compact Kähler surface admitting an Einstein metric g_e which is conformal to an extremal Kähler metric g_{ext} . As the metric g_e is Einstein we have the following Allendorfer-Weil formula

$$\chi(M) = \frac{1}{8\pi^2} \int_M |W_e|^2 + \frac{R_e^2}{24} dV_{g_e}$$

where W_e is the Weyl tensor. Hence we have the following

$$R_e^2 Vol(M, g_e) = 192\pi^2 \chi(M) - 24 \int_M |W_e|^2 dV_{g_e}.$$

The integral

$$\mathcal{F} = \int_M |W_e|^2 dV_{g_e}$$

is a conformal invariant so we can compute it with respect to the extremal Kähler metric g_{ext} . The Hirzebruch signature formula,

$$\sigma(M) = \frac{1}{12\pi^2} \int_M |W^+|^2 - |W^-|^2 dV_g,$$

allows use to write \mathcal{F} as

$$\mathcal{F} = \int_M 2|W_{ext}^+|^2 dV_{g_{ext}} - 12\pi^2 \sigma(M)$$

where W^+ and W^- denote the self-dual and anti-self-dual parts of the Weyl tensor. A standard fact in Kähler geometry is that, for a Kähler metric,

$$|W^+|^2 = \frac{R^2}{24} \text{ and so in our case } \mathcal{F} = \int_M \frac{R_{ext}^2}{12} dV_{g_{ext}} - 12\pi^2 \sigma(M).$$

Putting this all together and using Proposition 1 we have:

Proposition 2 *The Gaussian density of a conformally Kähler, Einstein metric is given by*

$$\Theta(M) = \frac{3}{2e^2} (2\chi(M) + 3\sigma(M)) - \frac{2c_{min}}{(8\pi e)^2}$$

where c_{min} is the Calabi energy of the extremal metric g_{ext} .

The Euler characteristic of $\mathbb{CP}^2 \# 2\overline{\mathbb{CP}^2}$ is 5 and its signature is -1 and in [4] c_{min} is given by $32\pi^2 \min_{\mathbb{R}} f$, where f is the following function

$$f(x) = 3 \left(\frac{32 + 176x + 318x^2 + 280x^3 + 132x^4 + 32x^5 + 3x^6}{12 + 72x + 138x^2 + 120x^3 + 54x^4 + 12x^5 + x^6} \right).$$

This is minimised at $x \approx 0.95771$ and gives a value of 7.13647 making the density of the Chen-LeBrun-Weber metric 0.4552 to 4 decimal places.

2.2.2 The Page metric

LeBrun in [8] carries out the same analysis for the Page metric on $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$. He computes that, in this case, c_{min} is given by $96\pi^2 \min_{\mathbb{R}} h$, where h is given by

$$h(x) = \left(\frac{4 + 14x + 16x^2 + 3x^3}{x(6 + 6x + x^2)} \right).$$

The function h is minimised when $x \approx 2.183933$ giving a value of 2.72621. The Euler characteristic of $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ is 4 and its signature is 0; putting these values into Proposition 2 we obtain that the density of the Page metric is 0.5172 to 4 decimal places. This value agrees with the one quoted in [1].

2.3 Ricci-Solitons

A theorem of Perelman [10] states that on a compact manifold any shrinking soliton is a gradient soliton. The soliton potential function is the function that achieves the infimum in the definition of the ν -energy. At a gradient Ricci-soliton (g_{sol}, f) Headrick and Wiseman [6], following Perelman, consider the following two quantities

$$Z(\beta) := \frac{1}{(2\pi e)^n} \int_M e^{-\beta f} dV_{g_{sol}}, \quad S(\beta) := (1 - \beta \frac{d}{d\beta}) \log Z(\beta).$$

Perelman shows, by examining the PDEs that g_{sol} and f must satisfy, that at a Ricci-soliton, $\nu(g_{sol}) = S(1)$. This means that in order to compute the Gaussian density one only needs to be able to evaluate $Z(\beta)$ which means being able find coordinates where one knows both the potential function f and the volume element $dV_{g_{sol}}$. If the metric is a ‘toric-Kähler’ metric then such a convenient coordinate system exists as we shall outline below.

2.3.1 Computing the potential function

In the interest of brevity we list the important features of the toric framework as outlined in the survey article [5]. The soliton we refer to here is the Wang-Zhu soliton g_{WZ} .

- We work on a dense open subset of $\mathbb{CP}^2 \# 2\overline{\mathbb{CP}^2}$ diffeomorphic to $P^\circ \times \mathbb{T}^2$ where P° is the interior of the pentagon with vertices at $(-1,-1)$, $(1,-1)$, $(1,0)$, $(0,1)$ and $(-1,1)$. We work in coordinates $(x_1, x_2, \theta_1, \theta_2)$, the x_i being coordinates on the pentagon and the θ_i being coordinates on the torus.
- The volume form in these coordinates is $dx_1 \wedge dx_2 \wedge d\theta_1 \wedge d\theta_2$.
- The soliton is a gradient soliton. The soliton potential function f is of the form

$$f(x_1, x_2) = c_1 x_1 + c_2 x_2$$

for some constants c_1 and c_2 . In fact we can restrict to metrics invariant under the bilateral symmetry $x_1 \longleftrightarrow x_2$. The potential function then has the form

$$f(x_1, x_2) = c(x_1 + x_2).$$

- The constant c must satisfy that

$$\int_P x_1 e^{-c(x_1+x_2)} dx = \int_P x_2 e^{-c(x_1+x_2)} dx = 0.$$

The constraint on c can be also be seen as saying that c minimises the convex function $F(c) = \int_P e^{-c(x_1+x_2)} dx$ so there is a unique c satisfying the above constraint. Computing this integral we see that

$$F(c) = \frac{(e^{2c} - 2 + (1 - c)e^{-c})}{c^2}.$$

F is minimised at $c \approx -0.434748$ and this value of c agrees with the one found numerically by Headrick and Wiseman in [6].

2.3.2 The value of $\Theta(g_{WZ})$

We recall, for the gradient soliton g_{sol} with potential function f , the following quantities

$$Z(\beta) := \frac{1}{(2\pi e)^n} \int_M e^{-\beta f} dV_{g_{sol}}, \quad S(\beta) := (1 - \beta \frac{d}{d\beta}) \log Z(\beta).$$

The ν -energy is then $S(1)$.

By the discussion above we know that $\beta = 1$ minimises the function $\log Z(\beta)$. So we have

$$\nu = \log Z(1) = \log \left(\frac{1}{(2\pi e)^2} \int_M e^{-f} dV_{g_{sol}} \right) = \log \left(\int_P e^{-c(x_1+x_2)} dx \right) - 2.$$

This gives

$$\Theta(g_{WZ}) = \frac{\min_{\mathbb{R}} F}{e^2} = \frac{3.36094}{e^2} = 0.4549$$

to 4 decimal places. Again this is in agreement with the value found in [6].

2.3.3 The Koiso-Cao soliton g_{KC}

The same method can be used to compute the density of the Koiso-Cao soliton on $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$. Here the coordinates are on the trapezium T with vertices at $(2,-1)$, $(-1,2)$, $(-1,0)$ and $(0,-1)$. The soliton potential function h is of the form

$$h = c(x_1 + x_2)$$

where the constant c is the unique minimiser of the function

$$H(c) = \int_T e^{-c(x_1+x_2)} dx.$$

Evaluating, we have that

$$H(c) = \frac{(e^{2c} - e^{-c} - 3ce^{-c})}{c^2}.$$

This function is minimised at $c \approx 0.5276$ giving a value of 3.8266. The density of g_{KC} is

$$\Theta(g_{KC}) = \frac{\min_{\mathbb{R}} H}{e^2} = \frac{3.38266}{e^2} = 0.5179$$

to 4 decimal places. This agrees with the value quoted in [1].

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